SCATTERING
ON MULTI-SCALE ROUGH SURFACES

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Abstract We present a method to recover a fractal dimension of a multi-scale rough surface, the so-called correlation dimension, from the knowledge of the far-field scattered intensity. The results are validated by numerical experiments on Weierstrass-like surfaces.

Keywords: multi-scale surface, correlation dimension, Weierstrass profile.

Introduction

Although simple, fractal models have shown to be of particular relevance for the description of natural rough surfaces. An important issue in remote sensing is therefore to understand the impact of fractal characteristics on electromagnetic wave scattering. Recently, several simple models such as Weierstrass functions of fractional Brownian motion have been proposed to describe multi-scale rough surfaces and the diffracted field has been studied by means of the usual approximations (Kirchhoff approach [1-7], the Extended Boundary Condition Method [8],[9] or Integral Equation Method [10]). Some qualitative results relating the scattering amplitude to fractal dimensions of the surface have been exhibited but failed to give a precise and general way of computing the latter quantities.

In a previous paper, the authors proposed a method to compute at least one fractal dimension (the so-called correlation dimension) of a rough surface by the sole knowledge of the far-field intensity. The technique relies mainly on the small-perturbation method and its simple connection to Fourier analysis. We will here give an outline of this method and provide an illustration on the simple example of band-limited Weierstrass functions.

1. DESCRIPTION OF THE SURFACE

Very often the rough surfaces are modeled by stationary random processes with given covariance and distribution heights. We will, however, work in a deterministic setting for the sake of simplicity. The spirit of the method is essentially the same in the random case, but one has to cope with additional complications pertaining to estimation theory. Although the deterministic description might be less realistic, it provides a good starting point to experiment the method.

Let us consider an infinite surface, invariant along the z direction, whose height about some reference plane is given by $y = f(x)$. The profile will be set to zero outside some interval of interest, say $f(x) = 0$ for $|x| > L$, and of zero mean inside, $\int f = 0$. Note that there is no restriction in supposing the range of interest of finite extent in the scattering problem as long as the latter remains significantly greater than the illuminated region.

The two-points correlation function

$$C(\tau) = \frac{1}{2L} \int_{-L}^{+L} dx f(x) f(\tau + x)$$

describes the spatial variation of heights. The root mean square (RMS) height $\sigma = \sqrt{C(0)}$ quantifies the roughness of the surface.

Usually, the fractal nature of a profile is expressed in terms of its Hausdorff-Besicovitch dimension (e.g. [11]) or simply box counting dimension, which is obtained in the following way. Given a function $f(x)$, count how many boxes or balls of radius ϵ are necessary to cover its graph. Calling $N(\epsilon)$ the number thus obtained for the optimal covering, the Hausdorff-Besicovitch dimension D_{HB} is the exponent governing the growth of $N(\epsilon)$ at small scales:

$$N(\epsilon) \sim \epsilon^{-D_{HB}}, \ \epsilon \to 0.$$

The dimension D_{HB} can be seen as an indicator of the roughness of the surface. At the two extremes we have $D_{HB} = 1$ for smooth curves and $D_{HB} = 2$ for curves that fill the plane. A curve with $D_{HB} > 1$ is non-differentiable. If it is, however, Hölder continuous with some Hölder exponent β (i.e if $|f(x+\epsilon) - f(x)| \leq C|\epsilon|^\beta$ for some positive constants $C, \beta > 0$), then $D_{HB} \leq 2 - \beta$. Now there is a close relationship between the fractality of a graph as described by the Hausdorff-Besicovitch dimension and the small-scale behavior of its correlation function. Suppose the following asymptotic behavior holds for all fixed τ and for some $0 < d < 2$:

$$C(\tau + h) - C(\tau) \sim h^d, \ h \to 0.$$

Then it is easy to show that $D_{HB} \leq 2 - \frac{d}{2}$ (in the random case where $C(\tau) = \ < f(x+\tau)f(x) >$ is a "true" correlation function we have even the stronger statement $D_{HB} = 2 - \frac{d}{2}$; see [12]). Hence the fractality of a profile can be quantified by the regularity of its correlation function at small scale. In fact, the exponent d has been shown [13] to coincide with one of the so-called generalized wavelet dimensions [14] constructed by means of the continuous wavelet transform: the wavelet correlation dimension, to which we will simply refer as the correlation dimension.

The advantage of the correlation dimension is that it can be computed very easily using quadratic averages of the Fourier transform. If an asymptotic scaling of the type

$$\int_\alpha^{\lambda\alpha} |\hat{f}(\alpha')|^2 d\alpha' \sim \alpha^{-\gamma}, \ \alpha \to \infty \tag{1}$$

is observed for some fixed $\lambda > 1$, then

$$\gamma = d, \tag{2}$$

as was shown in [13]. In practice the surfaces of interest are never true fractals, since they must be at least differentiable for the electromagnetic boundary conditions to be well-defined. However, the scaling (1) will hold in some range of scales and the governing exponent will be identified with the correlation dimension.

2. THE ELECTROMAGNETIC SCATTERING PROBLEM

Now let us come to the physical problem. Suppose the surface described above separates vacuum from a perfectly conducting metal. The variable parameter here will be the maximum amplitude of the surface, so it is convenient to introduce the normalized function $h(x)$ and the RMS height σ defined by $f(x) = \sigma\ h(x)$. An s polarized beam, with frequency ω and harmonic time dependence $e^{-i\omega t}$ is impinging from the top. In the low frequency limit, a rough surface scattering problem can be solved with the help of the small perturbation method. Up to first order in σ, it is shown that the scattering amplitude at infinity can be expressed in terms of the Fourier transform of the profile [15]. Indeed, denoting by α and α' the x components of the wave-vectors of the incident and scattered plane waves, respectively, the scattering amplitude $r(\alpha, \alpha')$ writes

$$r(\alpha, \alpha') = -\delta(\alpha' - \alpha) + i\,\frac{\beta(\alpha)}{\pi}\,\sigma\,\hat{h}(\alpha' - \alpha) + o(\sigma)\,, \tag{3}$$

with $\beta = \sqrt{k^2 - \alpha^2}$ if $|\alpha| \leq k$ or $\beta = i\sqrt{\alpha^2 - k^2}$ if $|\alpha| > k$, and where δ denotes Dirac's distribution.
The problem and the notations are depicted on Fig. 1.

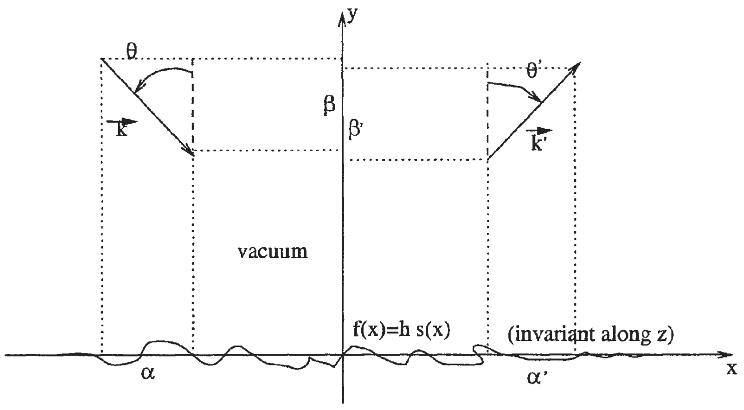

Figure 1. Description of the problem and notations

We will use formulae (1) and (2) to retrieve the correlation dimension of the surface from the scattering pattern. Numerical application will be performed on the so-called band-limited Weierstrass function:

$$h_N(x) = \frac{1}{\sigma_N} \sum_{n=1}^{N} a^n \cos\left(2\pi b^n x + \varphi_n\right). \tag{4}$$

Here a, b are two real positive numbers such that $a < 1, b > 1$ and $ab > 1$, and σ_N is chosen in such a way that $< h_N^2 >= 1$. The set $(\varphi_n)_n$ is a set of specified (arbitrary) phases. In some frequency range limited by the highest spatial frequency of h_N, the asymptotic of the Fourier transform of h_N is governed by the correlation dimension d of the true Weierstrass function h_∞ (that is the untroncated series). A simple computation gives

$$d = -2\frac{\log a}{\log b}\,. \tag{5}$$

Let us now consider an isotropic source, the so-called wire source in bidimensional problems, located at $\mathrm{S}(0, y_S)$, $y_S > 0$. The incident electric field writes, for $y < y_S$,

$$E^i(x,y) = E_0\; H_0^{(1)}(k\sqrt{x^2+(y-y_S)^2}) = E_0 \int_{-\infty}^{\infty} \frac{d\alpha}{\beta}\; e^{i\alpha x - i\beta(y-y_S)}\,, \tag{6}$$

E_0 being a complex constant, and $H_0^{(1)}$ the zero order Hankel function of the first kind. Assuming the source is far from the surface ($ky_S >> 1$), the integral can be restricted to propagating waves. Therefore, the incident field writes as a superposition of plane waves where each incidence angle θ is present, with amplitude $1/k\cos\theta$. Focusing on the backscattered field $E^b(S)$, we derive in the frame of the small perturbation method:

$$\begin{aligned} E^b(S) \;&\simeq E_0 \int_{-k}^{k} d\alpha\, \frac{r(\alpha,-\alpha)}{\beta}\; e^{2i\beta y_S} \\ &\simeq E_0\,(-e^{2iky_S} \;+\; \frac{i}{\pi}\int_{-k}^{k} d\alpha\, \sigma\hat{h}(-2\alpha)\; e^{2i\beta y_S})\,. \end{aligned} \tag{7}$$

The first term results from the specular reflection under normal incidence. Generally, remote sensing experiments avoid normal or near-normal incidences, because of the lack of information contained in the echo. In the same way, mathematically, such a term with constant modulus prevents any informative asymptotic behavior of the backscattered field. Therefore, in the following, we consider a wire source with some mask which stops the waves radiated around normal incidence. Denoting by $[-\theta_m, \theta_m]$ the darkened angular interval, and assuming that the contributions from the various incidence angles are not correlated in the far field, we obtain

$$I^b(k) = |E^b(S)|^2 = 2\left|\frac{E_0}{\pi}\right|^2 \int_{k\sin\theta_m}^{k} d\alpha\; |\sigma\hat{h}(-2\alpha)|^2\,. \tag{8}$$

According to (2), $I^b(k)$ behaves as k^{-d}. Consequently, the correlation dimension can be deduced from a few measurements.

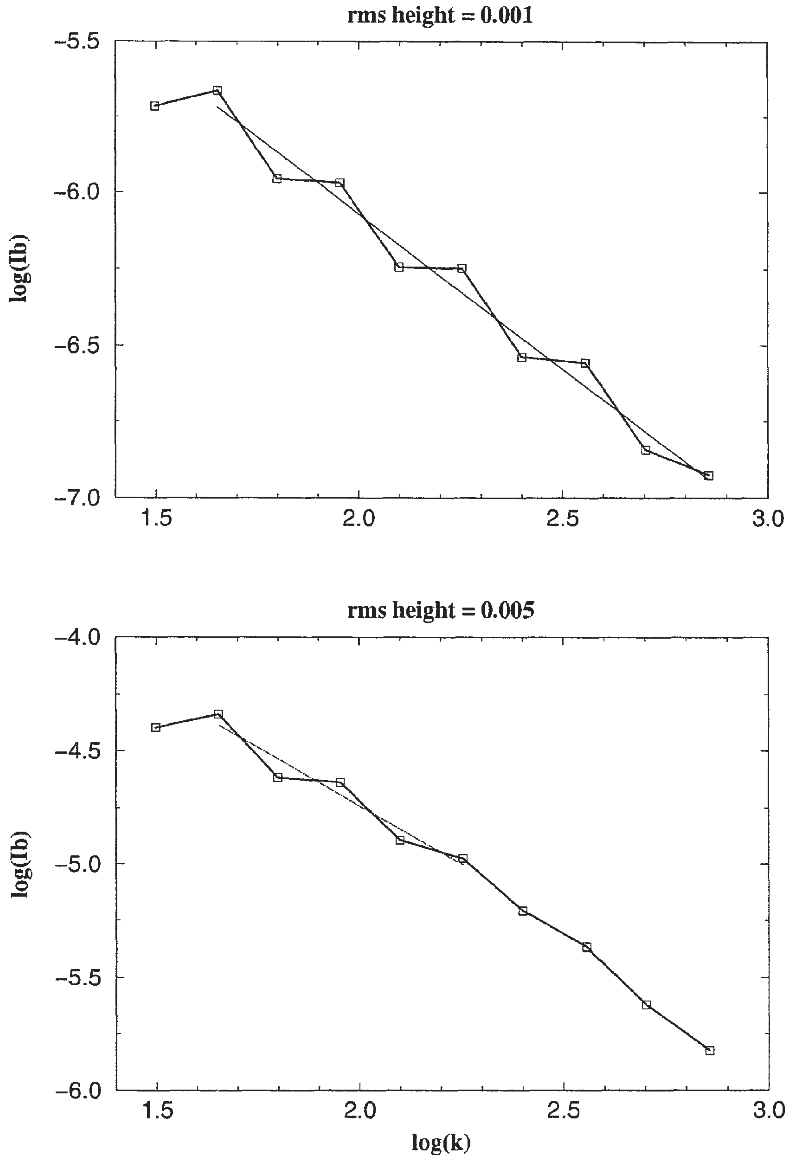

Figure 2. Back-scattered intensity from a Weierstrass surface for increasing RMS.

2.1. THE RIGOROUS SOLUTION

The rigorous solution of the scattering problem is achieved thanks to a classical boundary integral method, described in [16], coupled with a boundary finite-element method. We consider a band-limited Weierstrass function with $b = \sqrt{\pi}$, $a = 1/\sqrt{b}$ and $N = 9$. The ratio between the highest and the lowest spatial frequency is thus close to 100, and the correlation dimension $\kappa_2 = 1$. As a check, we have first computed the backscattered efficiencies for $\sigma = 10^{-3}$. With such a RMS height, when the wavenumber k increases from 2π to $2\pi b^8$, the ratio λ/σ decreases from 1000 to 20. In this range, the perturbation theory is supposed to give reliable results. As expected, a linear regression obtained from a log – log plot with nine values of the frequency gives a slope $\gamma = -1.01$ with a correlation coefficient $\rho = 0.99$, whereas the theoretical value is $\gamma = -\kappa_2 = -1$ [13]. Then, we have multiplied the RMS height by a factor of five, $\sigma = 5.10^{-3}$. Performing the linear regression on increasing ranges $\lambda/\sigma < 10$, $\lambda/\sigma < 4$ and $\lambda/\sigma < 1.2$, respectively, we obtain $\gamma = -1.03$, $\gamma = -1.08$ and $\gamma = -1.19$ respectively. This shows that the estimated dimension departs only slowly from the real one while increasing the ratio λ/σ, that is while leaving the domain of validity of the small perturbation method. The results are displayed on Fig. 2.

Conclusion

Relying upon the small-perturbation approximation, we have been able to recover in a simple way a fractal dimension of a rough surface. In addition, numerical experiments performed on the simple example of Weierstrass functions suggest that the method remains satisfactory far beyond the domain of validity of the small-perturbation. This is, however, only a first step. The correlation dimension alone is not sufficient to identify completely fractal surfaces, and a challenging problem for the future is to be able to recover the whole set of generalized (wavelet) fractal dimensions.

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